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Topological Charge of Lattice Abelian Gauge Theory

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ABSTRACT

Configuration space of abelian gauge theory on a periodic lattice becomes topologically disconnected by excising exceptional gauge field configurations. It is possible to define a $U(1)$ bundle from the nonexceptional link variables by a smooth interpolation of the transition functions. The lattice analogue of Chern character obtained by a cohomological technique based on the noncommutative differential calculus is shown to give a topological charge related to the topological winding number of the $U(1)$ bundle.

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1 Introduction

In a recent paper [1] Lüscher has investigated generic structures of chiral anomaly for abelian gauge theory on the lattice. His work is extended to arbitrary higher dimensions in our previous papers [2], where the topological part of the axial anomaly is shown to be interpretable as a lattice generalization of Chern character within the framework of noncommutative differential calculus [3]. In the continuum theory the Chern character gives the integer topological winding number when integrated over the base manifold and it coincides with the index of the Dirac operator [4]. The chiral anomaly on the lattice can also be related with the index [5] of the Ginsparg-Wilson Dirac operator [6, 7]. So it is very natural to expect some extension from the continuum to the lattice of the index theorem relating the analytical index of the Dirac operator with the topological invariant of the manifold on which the Dirac operator is defined. In this respect, it is, however, not clear in the constructions of ref. [1, 2] whether the lattice analogue of Chern character can be related to some topology of the gauge theory on the lattice.

One might think that it would make no sense to argue the topological configurations on the lattice since any lattice fields could be continuously deformed into the trivial configuration and no nontrivial topological invariants could be constructed. But this is not the case. As argued in refs. [8, 9, 10, 11], it is indeed possible to define a smooth fiber bundle and hence a topological winding number for a given lattice field configuration if it contains no exceptional link variables [8, 9, 11]. In the case of abelian theories Lüscher has shown in ref. [12] that the configuration space of the link variables satisfying the admissibility condition has a rich topological structure. It is considered as a kind of smoothness condition for the gauge field configuration, ensuring the existence of gauge potentials continuously parameterizing the link variables. The essential point here is that the configuration space of the admissible link variables is topologically disconnected.

In this paper we investigate the topological charge of abelian gauge theory on a periodic lattice in arbitrary even dimensions and argue that the lattice analogue of Chern character obtained in refs. [1, 2] indeed gives an integer-valued topological invariant by relating it to the topological winding number of a $U(1)$ bundle constructed from the lattice gauge fields by the interpolation method of refs. [8, 11].

We should add a brief argument concerning the theorem given in refs. [1, 2], where it is supposed for an infinite hypercubic regular lattice. We can extend it to topologically nontrivial lattices Λ without boundaries by restricting to ultralocal functions. We assume that Λ is locally hypercubic and regular. This means that for any point $n \in \Lambda$ one can find a set U_n of lattice points and links with a hypercubic regular lattice structure of the same dimensions. Hypercubic regular lattices with periodic boundary conditions, which we shall consider, are examples for Λ . We call functions f on Λ ultralocal if $f(n)$ for any $n \in \Lambda$

depends only on the gauge potentials associated to links within the subset U_n of Λ . The abelian gauge potentials on the lattice will be treated in Sect. 3 in detail. See also refs. [13, 1, 12]. Throughout this paper we assume the lattice spacing $a = 1$. The forward and backward difference operators Δ_μ and Δ_μ^* are then defined by

$$\Delta_\mu f(n) = f(n + \hat{\mu}) - f(n) , \quad \Delta_\mu^* f(n) = f(n) - f(n - \hat{\mu}) . \quad (1.1)$$

Then the theorem is extended to the lattice Λ as

Theorem : *Let q be gauge invariant and smooth ultralocal function of the abelian gauge potentials A_μ on a locally hypercubic regular lattice Λ of dimensions D without boundaries that satisfies the topological invariance*

$$\sum_{n \in \Lambda} \delta q(n) = 0 \quad (1.2)$$

for arbitrary local variations of the gauge potentials $A_\mu \rightarrow A_\mu + \delta A_\mu$, then $q(n)$ for arbitrary $n \in \Lambda$ takes the form

$$\begin{aligned} q(n) = & \sum_{l=0}^{[D/2]} \beta_{\mu_1 \nu_1 \dots \mu_l \nu_l} F_{\mu_1 \nu_1}(n) F_{\mu_2 \nu_2}(n + \mu_1 + \nu_1) \\ & \times \dots \times F_{\mu_l \nu_l}(n + \mu_1 + \nu_1 + \dots + \mu_{l-1} + \nu_{l-1}) + \Delta_\mu^* k_\mu(n) , \end{aligned} \quad (1.3)$$

where $F_{\mu\nu}(n) = \Delta_\mu A_\nu(n) - \Delta_\nu A_\mu(n)$ is the field strength, the coefficient $\beta_{\mu_1 \nu_1 \dots \mu_n \nu_n}$ is anti-symmetric in its indices and the current k_μ can be chosen to be gauge invariant and ultralocal in the gauge potential.

For functions q on the infinite lattice \mathbf{Z}^D the theorem holds true. Since Λ is assumed to be locally hypercubic and regular, the same identity should also follow for ultralocal functions q . The point here is that the gauge invariant current k_μ can be chosen to be ultralocal since both q and the topological terms are ultralocal and (1.3) must be an identity for any configuration of the gauge potentials. As we will see, the topological charge density in general has a complicated form. The theorem will be used to rewrite it to the standard form (1.3).

This paper is organized as follows. In the next section we describe the topological winding number of abelian gauge theory on a D dimensional torus, giving an explicit formula for the topological charge in terms of the transition functions. Sect. 3 deals with abelian gauge fields on a D dimensional periodic lattice. We argue the precise relation among the link variables, the field strengths and the gauge potentials on the lattice. In Sect. 4 we give a formulation of interpolation of the parallel transport functions from the discrete lattice to the continuum. The interpolated transition functions are obtained in closed form. In Sect. 5 we define the topological charge of the lattice abelian gauge theory by the Chern

number of the fiber bundle with the interpolated transition functions given in Sect. 4. The connection between the topological charge and the chiral anomaly obtained by the method of noncommutative differential calculus in our previous works is established. We also show that the topological charge can be expressed solely by the magnetic fluxes introduced in ref. [12]. A mathematical lemma concerning the existence of periodic potential functions for flux-free field configurations is given in Sect. 6. In the proof of the lemma we show the systematic way to isolate the flux contributions from the field strengths. For illustration we present a simple but nontrivial field configuration of constant field strengths with unit topological charges in two dimensions. Sect. 7 is devoted to summary and discussion.

2 Topological charge of abelian gauge theory on T^D

Fiber bundles over a manifold are topologically classified by the equivalence class of transition functions. Our first main concern is to give a formula for the topological winding number of the fiber bundle in terms of transition functions. In this paper we consider $U(1)$ bundles over a torus T^D of dimensions $D = 2N$ defined by the identification

$$x \sim x + L\hat{\mu} \quad \text{for } x \in \mathbf{R}^D, \mu = 1, \dots, D \quad (2.1)$$

where $\hat{\mu}$ denotes the unit vector in the μ -th direction and the period L of the torus is assumed to be a positive integer. A hypercubic periodic lattice Λ of dimensions D is defined as the set of integral lattice points in T^D .

We divide T^D into L^D hypercubes $c(n)$ ($n \in \Lambda$) defined by

$$c(n) = \{x \in T^D | x = n + \sum_{\mu=1}^D y_{\mu}\hat{\mu}, 0 \leq y_{\mu} \leq 1\} . \quad (2.2)$$

We assume that L is large enough so that any restricted bundle over $c(n)$ is trivial. Mathematically this can be achieved for $L \geq 2$. For later convenience, let us denote the intersection of $c(n)$ and $c(n - \hat{\mu})$ by $p(n, \mu)$ and the common boundary of $p(n, \mu)$, $p(n, \nu)$, \dots , $p(n, \sigma)$ by $p(n, \mu, \nu, \dots, \sigma)$.

Let $A^{(n)}(x)$ be the gauge potential 1-form[‡] on the hypercube $c(n)$, then the gauge potentials $A^{(n-\hat{\mu})}$ and $A^{(n)}$ are related by a gauge transformation on $p(n, \mu)$

$$A^{(n-\hat{\mu})}(x) = A^{(n)}(x) + d\Lambda_{n,\mu}(x) , \quad (2.3)$$

where d is the ordinary exterior differential and $\Lambda_{n,\mu}$ is defined by the transition functions $v_{n,\mu}$ as

$$v_{n,\mu} = e^{-i\Lambda_{n,\mu}} . \quad (2.4)$$

[‡]The gauge field $A_{\mu}^{(n)}(x)$ is assumed to be real.

The transition functions must satisfy cocycle conditions for $x \in p(n, \mu, \nu)$ [8]

$$v_{n-\hat{\mu}, \nu}(x) v_{n, \mu}(x) = v_{n-\hat{\nu}, \mu}(x) v_{n, \nu}(x) . \quad (2.5)$$

In terms of $\Lambda_{n, \mu}$ they can be written as

$$\Delta_{\mu}^* \Lambda_{n, \nu} = \Delta_{\nu}^* \Lambda_{n, \mu} \pmod{2\pi} , \quad (2.6)$$

where Δ_{μ}^* is the backward difference operator defined by (1.1).

The topological charge of the abelian gauge theory is given by

$$Q = C_N \int_{T^D} F^N , \quad C_N = \frac{1}{(2\pi)^N N!} , \quad (2.7)$$

where $F(x) = dA^{(n)}(x)$ is the field strength 2-form for $x \in c(n)$. It is gauge invariant and globally defined on T^D . By the Bianchi identity $dF = 0$, we have $F^N = d(A^{(n)} F^{N-1})$ and hence

$$\begin{aligned} Q &= C_N \sum_{n \in \Lambda} \int_{c(n)} d(A^{(n)} F^{N-1}) \\ &= C_N \sum_{n \in \Lambda} \sum_{\mu} \left(\int_{p(n+\mu, \mu)} A^{(n)} F^{N-1} - \int_{p(n, \mu)} A^{(n)} F^{N-1} \right) \\ &= C_N \sum_{n \in \Lambda} \sum_{\mu} \int_{p(n, \mu)} (A^{(n-\hat{\mu})} - A^{(n)}) F^{N-1} \\ &= C_N \sum_{n \in \Lambda} \sum_{\mu} \int_{p(n, \mu)} d\Lambda_{n, \mu} F^{N-1} . \end{aligned} \quad (2.8)$$

In the last step we have used the relation (2.3). Since $d\Lambda_{n, \mu}$ is a closed form, we can write $d\Lambda_{n, \mu} F^{N-1} = -d(d\Lambda_{n, \mu} A^{(n)} F^{N-2})$ and Q is further reduced as

$$Q = -C_N \sum_n \sum_{\mu, \nu}^{\sim} \int_{p(n, \mu, \nu)} \left\{ -\Delta_{\nu}^* d\Lambda_{n, \mu} A^{(n)} F^{N-2} + d\Lambda_{n-\hat{\nu}, \mu} d\Lambda_{n, \nu} F^{N-2} \right\} , \quad (2.9)$$

where $\sum_{\mu, \nu, \dots}^{\sim}$ implies antisymmetrized summation on μ, ν, \dots satisfying

$$\sum_{\mu, \nu, \dots}^{\sim} f_{\mu\nu\dots} = f_{12\dots} + \dots , \quad \sum_{\dots, \mu_i, \dots, \mu_j, \dots}^{\sim} = - \sum_{\dots, \mu_j, \dots, \mu_i, \dots}^{\sim} . \quad (2.10)$$

Since d commutes with Δ_{μ}^* and $d\Delta_{\nu}^* \Lambda_{n, \mu}$ is symmetric in μ and ν by the cocycle condition (2.6), the first term in the integrand of (2.9) does not contribute to Q . We thus obtain

$$Q = -C_N \sum_n \sum_{\mu, \nu}^{\sim} \int_{p(n, \mu, \nu)} d\Lambda_{n-\hat{\nu}, \mu} d\Lambda_{n, \nu} F^{N-2} . \quad (2.11)$$

Similar procedure can be repeated until all the field strengths disappear. The final expression for Q is given by

$$\begin{aligned}
Q &= (-1)^{\frac{N(N-1)}{2}} C_N \sum_n \sum_{\mu_1, \dots, \mu_N}^{\sim} \int_{p(n, \mu_1, \mu_2, \dots, \mu_N)} d\Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_N, \mu_1} d\Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_N, \mu_2} \cdots d\Lambda_{n, \mu_N} \\
&= (-1)^{\frac{N(N-1)}{2}} C_N \sum_n \epsilon_{\mu_1 \mu_2 \dots \mu_D} \int_{p(n, \mu_1, \mu_2, \dots, \mu_N)} d^N x \partial_{\mu_{N+1}} \Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_N, \mu_1} \partial_{\mu_{N+2}} \Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_N, \mu_2} \\
&\quad \times \cdots \times \partial_{\mu_D} \Lambda_{n, \mu_N} , \tag{2.12}
\end{aligned}$$

where $\epsilon_{\mu_1 \dots \mu_D}$ is the Levi-Civita symbol in D dimensions, ∂_μ stands for the derivative with respect to x_μ , and $d^N x$ is the volume form on $p(n, \mu_1, \dots, \mu_N)$.

To show that no new condition other than the cocycle conditions (2.6) is necessary we prove it by mathematical induction. Let us assume that

$$\begin{aligned}
Q &= (-1)^{\frac{k(k-1)}{2}} C_N \sum_n \sum_{\mu_1, \dots, \mu_k}^{\sim} \int_{p(n, \mu_1, \mu_2, \dots, \mu_k)} d\Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_k, \mu_1} d\Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_k, \mu_2} \\
&\quad \times \cdots \times d\Lambda_{n, \mu_k} F^{N-k} \tag{2.13}
\end{aligned}$$

holds true up to some integer $k < N$. Then by carrying out the manipulation from (2.8) to (2.9) we have

$$\begin{aligned}
Q &= (-1)^{\frac{k(k+1)}{2}} C_N \sum_n \sum_{\mu_1, \dots, \mu_{k+1}}^{\sim} \int_{p(n, \mu_1, \mu_2, \dots, \mu_k, \mu_{k+1})} \\
&\quad \times \left(-\Delta_{\mu_{k+1}}^* (d\Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_k, \mu_1} d\Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_k, \mu_2} \cdots d\Lambda_{n, \mu_k}) A^{(n)} \right. \\
&\quad \left. + d\Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_k-\hat{\mu}_{k+1}, \mu_1} d\Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_k-\hat{\mu}_{k+1}, \mu_2} \cdots d\Lambda_{n-\hat{\mu}_{k+1}, \mu_k} d\Lambda_{n, \mu_{k+1}} \right) F^{N-k-1} . \tag{2.14}
\end{aligned}$$

By noting the Leibnitz rule on the lattice $\Delta_\mu^*(f_n g_n) = \Delta_\mu^* f_n g_n + f_{n-\hat{\mu}} \Delta_\mu^* g_n$, we get

$$\begin{aligned}
&\Delta_{\mu_{k+1}}^* (d\Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_k, \mu_1} d\Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_k, \mu_2} \cdots d\Lambda_{n, \mu_k}) \\
&= d\Delta_{\mu_{k+1}}^* \Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_k, \mu_1} d\Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_k, \mu_2} \cdots d\Lambda_{n, \mu_k} \\
&\quad + d\Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_k-\hat{\mu}_{k+1}, \mu_1} d\Delta_{\mu_{k+1}}^* \Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_k, \mu_2} \cdots d\Lambda_{n, \mu_k} \\
&\quad + \cdots + d\Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_k-\hat{\mu}_{k+1}, \mu_1} d\Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_k-\hat{\mu}_{k+1}, \mu_2} \cdots d\Delta_{\mu_{k+1}}^* \Lambda_{n, \mu_k} . \tag{2.15}
\end{aligned}$$

The j -th term of the rhs of this expression is symmetric in μ_j and μ_{k+1} for $j = 1, \dots, k$ by the cocycle condition (2.6), hence the first term of the integrand of (2.14) does not contribute to the topological charge due to the antisymmetrized sum. The resulting expression for Q is just (2.13) with k replaced by $k+1$, implying that (2.13) holds true for any $k \leq N$. This completes the proof of (2.12).

3 Lattice abelian gauge field on the periodic lattice

The link variables $U_\mu(n)$ ($n \in \Lambda$, $\mu = 1, \dots, D$) are subject to the periodic boundary conditions

$$U_\mu(n + L\hat{\nu}) = U_\mu(n) , \quad (\nu = 1, \dots, D) , \quad (3.1)$$

and are assumed to be parametrized by

$$U_\mu(n) = e^{ia_\mu(n)} , \quad (-\pi \leq a_\mu(n) < \pi) , \quad (3.2)$$

where $a_\mu(n)$ is a vector field on Λ . The field strength is defined by

$$F_{\mu\nu}(n) = \frac{1}{i} \ln U_\mu(n) U_\nu(n + \hat{\mu}) U_\mu(n + \hat{\nu})^{-1} U_\nu(n)^{-1} , \quad |F_{\mu\nu}(n)| < \pi . \quad (3.3)$$

We exclude the exceptional field configurations [9] with $|F_{\mu\nu}(n)| = \pi$, where the plaquette variable equals -1 and $(-1)^y$ ($0 \leq y \leq 1$) becomes ambiguous.

The field strengths can be written in terms of a_μ as

$$F_{\mu\nu}(n) = \Delta_\mu a_\nu(n) - \Delta_\nu a_\mu(n) + 2\pi n_{\mu\nu}(n) , \quad (3.4)$$

where Δ_μ is the forward difference operator and $n_{\mu\nu}$ is an integer-valued anti-symmetric tensor field on Λ . The $n_{\mu\nu}(n)$ must be chosen so that the field strength $F_{\mu\nu}(n)$ lies within the principal branch of the logarithm in (3.3). Obviously, it satisfies $|n_{\mu\nu}(n)| \leq 2$. Furthermore, the link variables are assumed to be restricted so that the field strengths always satisfy the Bianchi identity $\Delta_{[\lambda} F_{\mu\nu]}(n) = 0$ in order to ensure the existence of gauge potential A_μ as

$$F_{\mu\nu}(n) = \Delta_\mu A_\nu(n) - \Delta_\nu A_\mu(n) . \quad (3.5)$$

This requirement is automatically satisfied by the admissibility condition of ref. [1, 12] that the field strengths satisfy

$$\sup_{n, \mu, \nu} |F_{\mu\nu}(n)| < \epsilon \quad (3.6)$$

for a fixed constant $0 < \epsilon < \pi/3$.[§] The exceptional field configurations mentioned above are also excluded by this condition.

Gauge transformations on Λ are defined by

$$U_\mu(n) \rightarrow V(n) U_\mu(n) V(n + \hat{\mu})^{-1} , \quad (3.7)$$

[§]As noted in ref. [1], ϵ should be replaced with ϵ/a^2 for the lattice spacing $a \neq 1$ and, hence, the restriction on the gauge field configuration disappears in the classical continuum limit.

where V is a $U(1)$ -valued function on Λ . The field strengths (3.3) is obviously gauge invariant. Note, however, that neither $\Delta_\mu a_\nu(n) - \Delta_\nu a_\mu(n)$ nor $n_{\mu\nu}(n)$ are separately gauge invariant. To see this we parameterize the gauge transformations V by a function λ on Λ as $V(n) = e^{i\lambda(n)}$, where λ is assumed to satisfy $\lambda(n + L\hat{\mu}) = \lambda(n)$ and $|\lambda(n)| \leq \pi$. Then $a_\mu(n)$ and $n_{\mu\nu}(n)$ are transformed as

$$a_\mu(n) \rightarrow a_\mu(n) - \Delta_\mu \lambda(n) + 2\pi N_\mu(n) , \quad n_{\mu\nu}(n) \rightarrow n_{\mu\nu}(n) - \Delta_\mu N_\nu(n) + \Delta_\nu N_\mu(n) , \quad (3.8)$$

where N_μ is an integer-valued vector field on Λ and must be chosen to satisfy

$$-\pi \leq a_\mu(n) - \Delta_\mu \lambda(n) + 2\pi N_\mu(n) < \pi . \quad (3.9)$$

If the field strength satisfies the Bianchi identities, so does $n_{\mu\nu}$. Hence it is always possible to find an integer-valued vector field m_μ on \mathbf{Z}^D satisfying

$$A_\mu(n) = a_\mu(n) + 2\pi m_\mu(n) , \quad \Delta_\mu m_\nu(n) - \Delta_\nu m_\mu(n) = n_{\mu\nu}(n) . \quad (3.10)$$

To show this we note that $m_\mu(n)$ is only determined up to integer-valued gauge transformations $m_\mu(n) \rightarrow m_\mu(n) - \Delta_\mu \Lambda(n)$ ($\Lambda(n) \in \mathbf{Z}$) on \mathbf{Z}^D and we can always work in the axial gauge $m_D(n) = 0$. In this gauge $m_\mu(n)$ ($\mu \neq D$) satisfies $\Delta_D m_\mu(n) = -n_{\mu D}(n)$, which can be integrated to

$$m_\mu(n) = - \sum_{y_D=0}^{n_D-1} n_{\mu D}(n_1, \dots, n_{D-1}, y_D) + m_\mu(n_1, \dots, n_{D-1}, 0) . \quad (3.11)$$

The sum on y_D in the rhs must make sense for arbitrary integer n_D . This can be achieved by defining the sum as

$$\sum_{y=a}^{b-1} f(y) = \begin{cases} f(a) + \dots + f(b-1) & (b > a) \\ 0 & (b = a) \\ -f(b) - \dots - f(a-1) & (b < a) \end{cases} \quad (3.12)$$

for arbitrary functions f on \mathbf{Z} and $a, b \in \mathbf{Z}$. It is just an analogue of one-dimensional integral on the discrete space \mathbf{Z} and satisfies the following properties

$$\begin{aligned} \sum_{y=a}^{b-1} f(y) &= - \sum_{y=b}^{a-1} f(y) , & \sum_{y=a}^{b-1} f(y) &= \sum_{y=a}^{c-1} f(y) + \sum_{y=c}^{b-1} f(y) , \\ \Delta_x \left(\sum_{y=a}^{x-1} f(y) \right) &= f(x) , & \sum_{y=a}^{b-1} \Delta_y f(y) &= f(b) - f(a) , \end{aligned} \quad (3.13)$$

where Δ_z stands for the difference operator with respect to $z \in \mathbf{Z}$. In (3.11) $m_\mu(n)|_{n_D=0} = m_\mu(n_1, \dots, n_{D-1}, 0)$ ($\mu \neq D$) are still to be determined. In order for (3.10) to be consistent they must satisfy the following set of equations

$$\Delta_\mu m_\nu(n_1, \dots, n_{D-1}, 0) - \Delta_\nu m_\mu(n_1, \dots, n_{D-1}, 0) = n_{\mu\nu}(n_1, \dots, n_{D-1}, 0) . \quad (3.14)$$

We see that the problem of finding $m_\mu(n)$ in D dimensions is reduced to the problem of solving the original equations dimensionally reduced to $D - 1$ dimensions. Again we can choose the axial gauge $m_{D-1}(n_1, \dots, n_{D-1}, 0) = 0$ and solve (3.14) as before. Obviously, this reduction process can be continued until all the m_μ 's are completely determined. We thus obtain

$$m_\mu(n) = - \sum_{\nu > \mu} \sum_{y_\nu=0}^{n_\nu-1} n_{\mu\nu}(n_1, \dots, n_{\nu-1}, y_\nu, 0, \dots, 0) . \quad (3.15)$$

It is integer-valued as announced.

From (3.10) the gauge potential A_μ also serves as the local coordinates for the link variables by the relations $U_\mu(n) = e^{iA_\mu(n)}$ [1, 12] and the gauge transformation (3.8) simply becomes

$$A_\mu(n) \rightarrow A_\mu(n) - \Delta_\mu \lambda(n) - 2\pi \Delta_\mu \Lambda(n) , \quad (3.16)$$

where Λ is an arbitrary integer-valued function on \mathbf{Z}^D . We emphasize that the gauge potential $A_\mu(n)$ defined by (3.10) and (3.15) is continuous at $U_\mu(n) = -1$ though $a_\mu(n)$ exhibits a discontinuity.

Note that the periodicity of the link variables and the field strengths only implies that of the gauge potentials up to gauge transformations by $2\pi\Lambda$ with $\Lambda(n) \in \mathbf{Z}$. In order to obtain nonvanishing topological charge it is necessary to have nonperiodic gauge potentials. For $m_\mu(n)$ given by (3.15) the periodicity of $A_\mu(n)$ is completely determined by $n_{\mu\nu}(n)$ as

$$\begin{aligned} A_\mu(n + L\hat{\nu}) - A_\mu(n) &= 2\pi m_\mu(n + L\hat{\nu}) - 2\pi m_\mu(n) \\ &= \begin{cases} -2\pi \sum_{n_\nu=0}^{L-1} n_{\mu\nu}(n_1, \dots, n_\nu, 0, \dots, 0) & \text{for } \mu < \nu \\ 0 & \text{for } \mu \geq \nu \end{cases} \end{aligned} \quad (3.17)$$

4 Interpolated transition functions

We now turn to the construction of the transition functions from the gauge fields on the lattice Λ by using the interpolation technique given in refs. [8, 11]. The main concerns of the authors of refs. [8, 10, 11] were the construction of the topological charges for non-abelian theories in four dimensions. The interpolated transition functions become more and more complicated as the dimensions increase in the nonabelian case so that the general expression cannot be available in arbitrary dimensions. In the abelian case, however, all the complications related to the noncommutativity of the transition functions[¶] disappear and it is possible to give transition functions in closed form as we will show.

[¶]In the nonabelian theories the transition functions are in fact matrix-valued in some representation.

We first define parallel transport functions $w^n(\bar{x})$ by

$$w^n(\bar{x}) = U_1(n)^{\sigma_1} U_2(n + \sigma_1 \hat{1})^{\sigma_2} \cdots U_D(n + \sigma_1 \hat{1} + \sigma_2 \hat{2} + \cdots + \sigma_{D-1} \widehat{D-1})^{\sigma_D} \quad (4.1)$$

for points \bar{x} of the corners of $c(n)$ given by

$$\bar{x} = n + \sum_{\mu=1}^D \sigma_\mu \hat{\mu}, \quad (\sigma_\mu = \{0, 1\}) . \quad (4.2)$$

At this point they are only defined on the corners of $c(n)$. In what follows we define iteratively interpolations of $w^n(\bar{x})$ over the boundary $\partial c(n)$.

Before going into mathematical detail, we mention the subtleties in defining arbitrary exponents $(w^n(\bar{x}))^y$ ($0 \leq y \leq 1$). We define exponents for link variables by $(U_\mu(n))^y \equiv e^{iy a_\mu(n)}$. But $(w^n(\bar{x}))^y$ does not coincide with $(U_1(n)^{\sigma_1})^y \cdots (U_D(n + \sigma_1 \hat{1} + \cdots + \sigma_{D-1} \widehat{D-1})^{\sigma_D})^y$ in general. The subtlety concerning the violation of such a naive distribution law of exponents can be avoided if we assume that $|a_\mu(n)|$ is sufficiently small for any link variable so that the naive distribution law is applicable. We assume this for the time being to justify the mathematical manipulations and remove such restrictions from the final expression of the transition functions.

Let $\mu_1, \dots, \mu_{D-1} = \{1, \dots, D\} \setminus \{\mu\}$ be the $D-1$ indices satisfying $\mu_1 < \cdots < \mu_{D-1}$. Then the interpolation along the $\hat{\mu}_{D-1}$ direction is defined by

$$\begin{aligned} w^m \left(n + \sum_{k=1}^{D-2} \sigma_k \hat{\mu}_k + y_{D-1} \hat{\mu}_{D-1} \right) &\equiv w^m \left(n + \sum_{k=1}^{D-2} \sigma_k \hat{\mu}_k + \hat{\mu}_{D-1} \right)^{y_{D-1}} \\ &\times w^m \left(n + \sum_{k=1}^{D-2} \sigma_k \hat{\mu}_k \right)^{1-y_{D-1}}, \end{aligned} \quad (4.3)$$

where $m = n$ or $m = n - \hat{\mu}$ and y_{D-1} ($0 \leq y_{D-1} \leq 1$) is the interpolation parameter and can be regarded as the coordinate of $c(n)$ in the $\hat{\mu}_{D-1}$ direction. We use (4.3) to define the second interpolation in the $\hat{\mu}_{D-2}$ direction as

$$\begin{aligned} &w^m \left(n + \sum_{k=1}^{D-3} \sigma_k \hat{\mu}_k + y_{D-2} \hat{\mu}_{D-2} + y_{D-1} \hat{\mu}_{D-1} \right) \\ &\equiv w^m \left(n + \sum_{k=1}^{D-3} \sigma_k \hat{\mu}_k + \hat{\mu}_{D-2} + y_{D-1} \hat{\mu}_{D-1} \right)^{y_{D-2}} w^m \left(n + \sum_{k=1}^{D-2} \sigma_k \hat{\mu}_k + y_{D-1} \hat{\mu}_{D-1} \right)^{1-y_{D-2}}, \end{aligned} \quad (4.4)$$

where y_{D-2} ($0 \leq y_{D-2} \leq 1$) is the new parameter for the interpolation.

We can carry out such interpolation procedure step by step until all the points in $p(n, \mu)$ are covered. After l steps, we obtain

$$w^m \left(n + \sum_{k=1}^{D-l-1} \sigma_k \hat{\mu}_k + \sum_{k=D-l}^{D-1} y_k \hat{\mu}_k \right)$$

$$\begin{aligned} &\equiv w^m \left(n + \sum_{k=1}^{D-l-1} \sigma_k \hat{\mu}_k + \hat{\mu}_{D-l} + \sum_{k=D-l+1}^{D-1} y_k \hat{\mu}_k \right)^{y_{D-l}} \\ &\quad \times w^m \left(n + \sum_{k=1}^{D-l-1} \sigma_k \hat{\mu}_k + \sum_{k=D-l+1}^{D-1} y_k \hat{\mu}_k \right)^{1-y_{D-l}} . \end{aligned} \quad (4.5)$$

In this way we can construct the parallel transport matrix $w^m(x)$ as

$$w^m(x) = \prod_{\{\sigma_k=0,1\}_{k=1,\dots,D-1}} w^m \left(n + \sum_{k=1}^{D-1} \sigma_k \hat{\mu}_k \right)^{\prod_{k=1}^{D-1} (\sigma_k y_k + (1-\sigma_k)(1-y_k))} \quad (4.6)$$

for any point $x \in p(n, \mu)$ given by

$$x = n + \sum_{k=1}^{D-1} y_k \hat{\mu}_k , \quad (0 \leq y_k \leq 1) . \quad (4.7)$$

In deriving (4.6) we have made a heavy use of the naive distribution law of exponents mentioned above. It is very important to note that the parallel transport function $\{w^n(x)\}_{x \in \partial c(n)}$ defines a continuous mapping $\partial c(n) \rightarrow U(1)$ as can be verified directly from (4.6).

In two and four dimensions $w^m(x)$ are explicitly given by

$$D = 2 : \quad w^m(x) = w^m(n)^{1-y_1} w^m(n + \hat{\mu}_1)^{y_1} \quad (4.8)$$

$$\begin{aligned} D = 4 : \quad w^m(x) &= w^m(n)^{(1-y_1)(1-y_2)(1-y_3)} w^m(n + \hat{\mu}_1)^{y_1(1-y_2)(1-y_3)} w^m(n + \hat{\mu}_2)^{(1-y_1)y_2(1-y_3)} \\ &\quad \times w^m(n + \hat{\mu}_3)^{(1-y_1)(1-y_2)y_3} w^m(n + \hat{\mu}_2 + \hat{\mu}_3)^{(1-y_1)y_2y_3} \\ &\quad \times w^m(n + \hat{\mu}_1 + \hat{\mu}_3)^{y_1(1-y_2)y_3} w^m(n + \hat{\mu}_1 + \hat{\mu}_2)^{y_1y_2(1-y_3)} \\ &\quad \times w^m(n + \hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3)^{y_1y_2y_3} \end{aligned} \quad (4.9)$$

Following ref. [11], we define the transition functions $v_{n,\mu}(x)$ interpolated over $p(n, \mu)$ by

$$v_{n,\mu}(x) \equiv w^{n-\hat{\mu}}(x) w^n(x)^{-1} . \quad (4.10)$$

With this definition one can easily show that the transition functions indeed satisfy the cocycle conditions (2.5). It is not so difficult to compute the transition functions explicitly in arbitrary dimensions. We first note that the interpolated transition functions $v_{n,\mu}(x)$ ($x \in p(n, \mu)$) are given by

$$v_{n,\mu}(x) = \prod_{\{\sigma_k=0,1\}_{k=1,\dots,D-1}} v_{n,\mu} \left(n + \sum_{k=1}^{D-1} \sigma_k \hat{\mu}_k \right)^{\prod_{k=1}^{D-1} (\sigma_k y_k + (1-\sigma_k)(1-y_k))} . \quad (4.11)$$

Again use has been made of the naive manipulation on the exponents. For the corner points of $p(n, \mu)$ it is straightforward to show that the transition functions are given by

$$v_{n,\mu} \left(n + \sum_{k=1}^{D-1} \sigma_k \hat{\mu}_k \right) = v_{n,\mu}(n) \exp \left[i \sum_{\substack{k=1 \\ \mu_k < \mu}}^{D-1} \sigma_k F_{\mu_k \mu} (n - \hat{\mu} + \sigma_1 \hat{\mu}_1 + \dots + \sigma_{k-1} \hat{\mu}_{k-1}) \right] , \quad (4.12)$$

where the field strengths are given by (3.3) and $v_{n,\mu}(n) = w^{n-\hat{\mu}}(n)w^n(n)^{-1} = U_\mu(n - \hat{\mu})$ is consistent with (4.10). Putting this into the rhs of (4.11), we obtain $v_{n,\mu}(x)$ explicitly as

$$v_{n,\mu}(x) = v_{n,\mu}(n) \exp \left[i \sum_{\substack{k=1 \\ \mu_k < \mu}}^{D-1} \sum_{\{\sigma_j=0,1\}_{j=1,\dots,k-1}} F_{\mu_k \mu} \left(n - \hat{\mu} + \sum_{l=1}^{k-1} \sigma_l \hat{\mu}_l \right) \right. \\ \left. \times \prod_{l=1}^{k-1} (\sigma_l y_l + (1 - \sigma_l)(1 - y_l)) y_k \right] \quad (4.13)$$

So far we have assumed that $|a_\mu(n)|$ is sufficiently small for any link variable so that all the mathematical manipulations concerning the distribution law of exponents can be justified. The final expression (4.13), however, satisfies all the desired properties expected for the transition functions such as the gauge covariance and the cocycle conditions even for general configurations as far as the exceptional configurations [9] containing $F_{\mu\nu}(n) = \pm\pi$ for some μ, ν and n is excluded and the field strengths satisfy the Bianchi identities. Henceforth, we consider general field configurations not necessarily restricted to be small and regard (4.13) as the definition of the transition functions.

In the classical continuum limit we may retain only the leading terms as

$$F_{\mu_k \mu} \left(n - \hat{\mu} + \sum_{l=1}^{k-1} \sigma_l \hat{\mu}_l \right) = F_{\mu_k \mu}(n) + \dots \quad (4.14)$$

The transition function (4.13) reduces to a simple form in the classical continuum limit as

$$v_{n,\mu}(x) = v_{n,\mu}(n) \exp \left[i \sum_{\substack{k=1 \\ \mu_k < \mu}}^{D-1} F_{\mu_k \mu}(n) y_k + \dots \right]. \quad (4.15)$$

Concrete expressions of the transition functions in some lower dimensions can be easily found from (4.13);

$D = 2 :$

$$\begin{aligned} v_{n,1}(x) &= v_{n,1}(n), & (x \in p(n, 1)), \\ v_{n,2}(x) &= v_{n,2}(n) e^{iy_1 F_{12}(n-\hat{2})}, & (x \in p(n, 2)), \end{aligned} \quad (4.16)$$

$D = 4 :$

$$\begin{aligned} v_{n,1}(x) &= v_{n,1}(n), & (x \in p(n, 1)), \\ v_{n,2}(x) &= v_{n,2}(n) e^{iy_1 F_{12}(n-\hat{2})}, & (x \in p(n, 2)), \\ v_{n,3}(x) &= v_{n,3}(n) e^{i(y_1 F_{13}(n-\hat{3}) + (1-y_1)y_2 F_{23}(n-\hat{3}) + y_1 y_2 F_{23}(n+\hat{1}-\hat{3}))}, & (x \in p(n, 3)), \\ v_{n,4}(x) &= v_{n,4}(n) \exp [y_1 F_{14}(n-\hat{4}) + (1-y_1)y_2 F_{24}(n-\hat{4}) + (1-y_1)(1-y_2)y_3 F_{34}(n-\hat{4}) \\ &\quad + y_1 y_2 F_{24}(n+\hat{1}-\hat{4}) + y_1(1-y_2)y_3 F_{34}(n+\hat{1}-\hat{4}) \\ &\quad + (1-y_1)y_2 y_3 F_{24}(n+\hat{2}-\hat{4}) + y_1 y_2 y_3 F_{34}(n+\hat{1}+\hat{2}-\hat{4})], \\ & & (x \in p(n, 4)). \end{aligned} \quad (4.17)$$

It is instructive to see in four dimensions that the Bianchi identities is necessary for the cocycle conditions (2.5) to be fulfilled. Let us consider the case $\mu = 2$ and $\nu = 3$, then we have for $x \in p(n, 2, 3)$

$$\begin{aligned} v_{n,2}(x) &= v_{n,2}(n)e^{iy_1 F_{12}(n-\hat{2})} , & v_{n,3}(x) &= v_{n,3}(n)e^{iy_1 F_{13}(n-\hat{3})} , \\ v_{n-\hat{3},2}(x) &= v_{n,2}(x)|_{y_3=1, n \rightarrow n-\hat{3}} = v_{n-\hat{3},2}(n-\hat{3})e^{iy_1 F_{12}(n-\hat{2}-\hat{3})} , \\ v_{n-\hat{2},3}(x) &= v_{n,3}(x)|_{y_2=1, n \rightarrow n-\hat{2}} = v_{n-\hat{2},3}(n-\hat{2})e^{i(y_1 F_{13}(n-\hat{2}-\hat{3})+(1-y_1)F_{23}(n-\hat{2}-\hat{3})+y_1 F_{23}(n+\hat{1}-\hat{2}-\hat{3}))} . \end{aligned} \quad (4.18)$$

The cocycle condition in the present case follows from the relations

$$\begin{aligned} v_{n-\hat{2},3}(n-\hat{2})v_{n,2}(n)e^{iF_{23}(n-\hat{2}-\hat{3})} &= v_{n-\hat{3},2}(n-\hat{3})v_{n,3}(n) , \\ \Delta_{[1}F_{23]}(n-\hat{2}-\hat{3}) &= 0 . \end{aligned} \quad (4.19)$$

5 Topological charge of lattice abelian gauge theory on the periodic lattice

We have constructed a set of transition functions (4.13) from the link variables on the lattice Λ . The transition functions in turn define a fiber bundle over T^D , for which the topological charge Q can be computed unambiguously by (2.12). In order to compute Q , let us define 1-form $d\Lambda_{n,\mu}(x)$ on $p(n, \mu)$ by

$$d\Lambda_{n,\mu}(x) = -i v_{n,\mu}(x)dv_{n,\mu}(x)^{-1} , \quad (5.1)$$

where d is the ordinary exterior derivative with respect to the continuous coordinates x_μ . Then Q is given by

$$Q = \sum_{n \in \Lambda} q(n) , \quad (5.2)$$

where q is the topological charge density satisfying $q(n + L\hat{\mu}) = q(n)$ for $\mu = 1, \dots, D$ and can be chosen to be

$$\begin{aligned} q(n) &= (-1)^{\frac{N(N-1)}{2}} C_N \epsilon_{\mu_1 \mu_2 \dots \mu_D} \int_{p(n, \mu_1, \mu_2, \dots, \mu_N)} d^N x \partial_{\mu_{N+1}} \Lambda_{n-\hat{\mu}_2-\dots-\hat{\mu}_N, \mu_1} \partial_{\mu_{N+2}} \Lambda_{n-\hat{\mu}_3-\dots-\hat{\mu}_N, \mu_2} \\ &\quad \times \dots \times \partial_{\mu_D} \Lambda_{n, \mu_N} . \end{aligned} \quad (5.3)$$

From (4.13) and (5.1) one can see that $q(n)$ is invariant under the gauge transformations (3.16), ultralocal in the gauge potential and a sum of products of N field strengths. Furthermore, the classical continuum limit can be obtained from (4.15) as

$$q(n) = \frac{1}{2^N} C_N \epsilon_{\mu_1 \nu_1 \dots \mu_N \nu_N} F_{\mu_1 \nu_1}(n) \dots F_{\mu_N \nu_N}(n) + \dots . \quad (5.4)$$

These properties together with the topological invariance of Q , i.e., the invariance under arbitrary local variations of the gauge potential as in (1.2), which is obvious by construction, in fact determine the structure of $q(n)$ by the theorem (1.3). In the present case we have $\beta_{\mu_1\nu_1\cdots\mu_l\nu_l} = 0$ for $l < N$ and $\beta_{\mu_1\nu_1\cdots\mu_N\nu_N} = C_N/2^N \epsilon_{\mu_1\nu_1\cdots\mu_N\nu_N}$ as can be seen from (5.4). Furthermore, the current divergence term in (1.3) does not contribute to the topological charge Q due to the periodic boundary conditions. We thus obtain the topological charge

$$Q = \frac{1}{2^N} C_N \sum_{n \in \Lambda} \epsilon_{\mu_1\nu_1\cdots\mu_N\nu_N} F_{\mu_1\nu_1}(n) F_{\mu_2\nu_2}(n + \hat{\mu}_1 + \hat{\nu}_1) \\ \times \cdots \times F_{\mu_N\nu_N}(n + \hat{\mu}_1 + \hat{\nu}_1 + \cdots + \hat{\mu}_{N-1} + \hat{\nu}_{N-1}) . \quad (5.5)$$

The summand of this expression is nothing but the Chern character obtained solely from the noncommutative differential calculus on the lattice. This establishes the connection between the lattice analogue of the Chern character and the topology of the fiber bundle over the discrete lattice as in continuum theory.

It is possible to express the topological charge in terms of the magnetic fluxes $\phi_{\mu\nu}(x)$ through the $\mu\nu$ -plane [12]. They are defined by

$$\phi_{\mu\nu}(n) = \sum_{s,t=0}^{L-1} F_{\mu\nu}(n + s\hat{\mu} + t\hat{\nu}) \\ = 2\pi \sum_{s,t=0}^{L-1} n_{\mu\nu}(n + s\hat{\mu} + t\hat{\nu}) , \quad (5.6)$$

where use has been made of (3.4). We immediately see that they are gauge invariant and integer multiples of 2π . Furthermore, they can be shown to be constants by virtue of the Bianchi identities [12]. Since the field strengths are continuous with respect to continuous changes of link variables as far as the exceptional gauge field configurations are excluded, the fluxes $\phi_{\mu\nu}$ must be invariant under such continuous changes of link variables. This implies that the sets of link variables $\{U_\mu(n)\}_{n \in \Lambda, \mu=1,\dots,D}$ with distinct fluxes $\{\phi_{\mu\nu}\}_{\mu,\nu=1,\dots,D}$ are topologically disjoint one another.

For later convenience, we introduce a set of integers $m_{\mu\nu} = -m_{\nu\mu}$ by

$$m_{\mu\nu} = \sum_{s,t=0}^{L-1} n_{\mu\nu}(n + s\hat{\mu} + t\hat{\nu}) . \quad (5.7)$$

If we define $\rho_{\mu\nu}(n)$ by

$$\rho_{\mu\nu}(n) = 2\pi \left(n_{\mu\nu}(n) - \frac{1}{L^2} m_{\mu\nu} \right) , \quad (5.8)$$

then we have

$$\sum_{s,t=0}^{L-1} \rho_{\mu\nu}(n + s\hat{\mu} + t\hat{\nu}) = 0 . \quad (5.9)$$

Furthermore, $\rho_{\mu\nu}(n)$ satisfies the Bianchi identities. Now using the lemma given in the next section, it is always possible to find a periodic vector field λ_μ on Λ satisfying

$$\Delta_\mu \lambda_\nu(n) - \Delta_\nu \lambda_\mu(n) = \rho_{\mu\nu}(n) . \quad (5.10)$$

This implies that the field strength $F_{\mu\nu}(n)$ can always be written as

$$F_{\mu\nu}(n) = \Delta_\mu \tilde{a}_\nu(n) - \Delta_\nu \tilde{a}_\mu(n) + \frac{2\pi}{L^2} m_{\mu\nu} , \quad (5.11)$$

where we have introduced a vector field on Λ by $\tilde{a}_\mu(n) \equiv a_\mu(n) + \lambda_\mu(n)$. It is possible to find a gauge potential $\tilde{A}_\mu(n)$ in terms of $\tilde{a}_\mu(n)$ and $m_{\mu\nu}$ as

$$\tilde{A}_\mu(n) = \tilde{a}_\mu(n) - \frac{2\pi}{L^2} \sum_{\nu > \mu} m_{\mu\nu} n_\nu . \quad (5.12)$$

This satisfies the periodicity differing from (3.17) as

$$\tilde{A}_\mu(n + L\hat{\nu}) - \tilde{A}_\mu(n) = \begin{cases} -2\pi m_{\mu\nu}/L & (\mu < \nu) \\ 0 & (\mu \geq \nu) \end{cases} . \quad (5.13)$$

Putting (5.11) into (5.5) and noting that the periodic vector field $\tilde{a}_\mu(n)$ does not contribute to the topological charge, we obtain

$$Q = \frac{1}{2^N N!} \epsilon_{\mu_1 \nu_1 \dots \mu_N \nu_N} m_{\mu_1 \nu_1} \dots m_{\mu_N \nu_N} , \quad (5.14)$$

where use has been made of the explicit form of C_N given in (2.7). The rhs of this expression is manifestly an integer and topological invariant as it should be.

6 Lemma

In this section we describe the Lemma used in the previous section. It turns out to be useful in finding link variables for given field strengths. We now state the lemma in the following form.

Lemma: *Let $\rho_{\mu\nu}$ be an antisymmetric tensor field on the D dimensional periodic lattice Λ satisfying*

$$\begin{aligned} \Delta_{[\lambda} \rho_{\mu\nu]}(n) &= 0 , & \rho_{\mu\nu}(n + L\hat{\lambda}) &= \rho_{\mu\nu}(n) , \\ \sum_{s,t=0}^{L-1} \rho_{\mu\nu}(n + s\hat{\mu} + t\hat{\nu}) &= 0 , & (\lambda, \mu, \nu &= 1, \dots, D) \end{aligned} \quad (6.1)$$

then there exists a vector fields λ_μ satisfying

$$\Delta_\mu \lambda_\nu(n) - \Delta_\nu \lambda_\mu(n) = \rho_{\mu\nu}(n) , \quad \lambda_\mu(n + L\hat{\nu}) = \lambda_\mu(n) , \quad (\mu, \nu = 1, \dots, D) . \quad (6.2)$$

Proof: The proof will proceed as in the case of solving (3.10) for $m_\mu(n)$. In the present case $\lambda_\mu(n)$ must be periodic in the lattice coordinates. This makes the solution somewhat complicated.

We first assume the existence of λ_μ satisfying (6.2) and note that the Bianchi identity and the periodicity in (6.1) imply the relation

$$\Delta_\mu \left(\sum_{s=0}^{L-1} \rho_{\lambda\nu}(n + s\hat{\lambda}) \right) = \Delta_\nu \left(\sum_{s=0}^{L-1} \rho_{\lambda\mu}(n + s\hat{\lambda}) \right) . \quad (6.3)$$

Now let us denote a sum of vector field V_μ along a lattice path C from $a \in \Lambda$ to $b \in \Lambda$ through the lattice points $a + \hat{\mu}$, $a + \hat{\mu} + \hat{\nu}$, \dots , $b - \hat{\rho}$ in this order by

$$\sum_C V_\mu(y) \Delta y_\mu = V_\mu(a) + V_\nu(a + \hat{\mu}) + \dots + V_\rho(b - \hat{\rho}) , \quad (6.4)$$

then (6.3) guarantees that the vector field α_λ defined by the sum along any lattice path C from $y = 0$ to $y = n$

$$\alpha_\lambda(n) = \alpha_\lambda(0) + \sum_C \sum_{s=0}^{L-1} \rho_{\lambda\mu}(y + s\hat{\lambda}) \Delta y_\mu \quad (6.5)$$

is independent of the path chosen. It is also periodic as can be seen from the third property of (6.1). From (6.2) and (6.5) we get

$$\Delta_\mu \left(\sum_{s=0}^{L-1} \lambda_\nu(n + s\hat{\nu}) \right) = \sum_{s=0}^{L-1} \rho_{\mu\nu}(n + s\hat{\nu}) = -\Delta_\mu \alpha_\nu(n) . \quad (6.6)$$

Hence we may choose $\alpha_\mu(n)$ without loss of generality to satisfy

$$\alpha_\mu(n) = - \sum_{s=0}^{L-1} \lambda_\mu(n + s\hat{\mu}) \quad (6.7)$$

We next define $\psi_\mu(n)$ and $\omega_{\mu\nu}(n)$ by

$$\psi_\mu(n) = \lambda_\mu(n) + \frac{1}{L} \alpha_\mu(n) , \quad \omega_{\mu\nu}(n) = \rho_{\mu\nu}(n) + \frac{1}{L} (\Delta_\mu \alpha_\nu(n) - \Delta_\nu \alpha_\mu(n)) . \quad (6.8)$$

From (6.2) and (6.7) one can easily show that these fulfill the following relations

$$\begin{aligned} \Delta_\mu \psi_\nu(n) - \Delta_\nu \psi_\mu(n) &= \omega_{\mu\nu}(n) , & \Delta_{[\lambda} \omega_{\mu\nu]}(n) &= 0 , \\ \sum_{s=0}^{L-1} \psi_\mu(n + s\hat{\mu}) &= \sum_{s=0}^{L-1} \omega_{\mu\nu}(n + s\hat{\mu}) = 0 . \end{aligned} \quad (6.9)$$

It is now possible to solve these with respect to $\psi_\mu(n)$ just like (3.10) by working with the axial gauge $\psi_D(n) = 0$. In this gauge $\psi_\mu(n)$ ($\mu = 1, \dots, D-1$) are given by

$$\psi_\mu(n) = - \sum_{y_D=0}^{n_D-1} \omega_{\mu D}(n_1, \dots, n_{D-1}, y_D) + \psi_\mu(n_1, \dots, n_{D-1}, 0) , \quad (6.10)$$

where $\psi_\mu(n_1, \dots, n_{D-1}, 0)$ is periodic in the lattice coordinates and still to be determined. It must satisfy

$$\begin{aligned} \Delta_\mu \psi_\nu(n_1, \dots, n_{D-1}, 0) - \Delta_\nu \psi_\mu(n_1, \dots, n_{D-1}, 0) &= \omega_{\mu\nu}(n_1, \dots, n_{D-1}, 0) , \\ \sum_{n_\mu=0}^{L-1} \psi_\mu(n_1, \dots, n_{D-1}, 0) &= 0 . \end{aligned} \quad (6.11)$$

These are the equations (6.9) restricted to $n_D = 0$, and can be solved again by choosing $\psi_{D-1}(n_1, \dots, n_{D-1}, 0) = 0$. This procedure can be continued until all the $\psi_\mu(n)$ are found. This completes the proof of the existence of ψ_μ and, hence, λ_μ by (6.8).

To illustrate all these ideas in a concrete example, let us consider a constant field^{||} $F_{\mu\nu}(n) = \epsilon_{\mu\nu} B$ in two dimensions.** If we parameterize the link variables as in (2.3), $a_\mu(n)$ ($\mu = 1, 2$) satisfy

$$F_{12}(n) = \Delta_1 a_2(n) - \Delta_2 a_1(n) + 2\pi n_{12}(n) = B , \quad (-\pi \leq a_\mu(n) < \pi , \quad |n_{12}(n)| \leq 2) . \quad (6.12)$$

The magnetic flux $\phi \equiv 2\pi\phi_{12}$ is given by

$$\phi = BL^2 = \sum_{n \in \Lambda} 2\pi n_{12}(n) . \quad (6.13)$$

This implies that BL^2 must be an integer multiple of 2π . For $BL^2 = 2\pi$ we may choose $n_{\mu\nu}(n)$ as

$$n_{\mu\nu}(n) = \delta_{\tilde{n}_1, [L/2]} \delta_{\tilde{n}_2, [L/2]} , \quad (6.14)$$

where we have introduced periodic lattice coordinates

$$\tilde{n}_\mu = n_\mu - L\epsilon\left(\frac{n_\mu}{L} + \frac{1}{2}\right) , \quad ([-L/2] < \tilde{n}_\mu \leq [L/2]) \quad (6.15)$$

with $\epsilon(x)$ being the stair-step function defined by

$$\epsilon(x) = n \quad \text{for} \quad n < x \leq n+1 , \quad (n \in \mathbf{Z}) . \quad (6.16)$$

^{||}General constant fields of arbitrary magnetic fluxes are explicitly given in ref. [12].

^{**}In two dimensions the Bianchi identities $\Delta_{[\lambda} F_{\mu\nu]} = 0$ are trivially satisfied for any anti-symmetric tensor fields.

The $a_\mu(n)$'s now satisfy the following equation

$$\Delta_1 a_2(n) - \Delta_2 a_1(n) = B - 2\pi n_{12}(n) = B - 2\pi \delta_{\tilde{n}_1, [L/2]} \delta_{\tilde{n}_2, [L/2]} . \quad (6.17)$$

We can solve this equation by following the procedure described in the proof of the lemma. We thus obtain

$$a_1(n) = -\frac{2\pi}{L} \tilde{n}_2 \delta_{\tilde{n}_1, [L/2]} , \quad a_2(n) = \frac{2\pi}{L^2} \tilde{n}_1 . \quad (6.18)$$

The gauge potential $A_\mu(n)$ can be found by solving (3.10) in the present case. The $m_\mu(n)$'s are easily obtained from (3.15) as

$$m_1(n) = -\delta_{\tilde{n}_1, [L/2]} \epsilon \left(\frac{n_2}{L} + \frac{1}{2} \right) , \quad m_2(n) = 0 . \quad (6.19)$$

In deriving this use has been made of the relation

$$\Delta_\mu \tilde{n}_\mu = 1 - L \delta_{\tilde{n}_\mu, [L/2]} , \quad (\text{no sum on } \mu) . \quad (6.20)$$

We thus find the gauge potential as

$$A_1(n) = a_1(n) + 2\pi m_1(n) = -LB \delta_{\tilde{n}_1, [L/2]} n_2 , \quad A_2(n) = B \tilde{n}_1 . \quad (6.21)$$

The nontrivial periodicity property of the gauge potential is

$$A_1(n + L\hat{2}) - A_1(n) = -L^2 B \delta_{\tilde{n}_1, [L/2]} = -2\pi \delta_{\tilde{n}_1, [L/2]} . \quad (6.22)$$

It is also possible to write the gauge potential in the form (5.12) as

$$\tilde{A}_1(n) = -\frac{2\pi}{L^2} n_2 = -B n_2 , \quad \tilde{A}_2(n) = 0 . \quad (6.23)$$

Finally the topological charge is given by

$$Q = \frac{1}{4\pi} \sum_{n \in \Lambda} \epsilon_{\mu\nu} F_{\mu\nu}(n) = \frac{L^2 B}{2\pi} = 1 . \quad (6.24)$$

7 Summary and discussion

We have argued that the topological charge obtained from the lattice generalization of the Chern character is indeed an integer related to the winding number of a U(1) bundle constructed from the link variables by a smooth interpolation. It picks up the topological structure of the underlying lattice originating from the periodicity. The configuration space of the link variables that is topologically trivial and connected to the trivial one becomes topologically disjoint by excising the exceptional field configurations. No two belonging

to different connected components can be continuously deformed into each other without crossing the exceptional configurations where the topological charge may jump due to the discontinuities of the field strengths at the exceptional plaquette variables. Link variables with different magnetic fluxes belong to different connected components of the field configurations. We have established an explicit relation between the topological charge and the magnetic fluxes.

The construction presented in this paper is rather indirect. One might think that the interpolation to a smooth fiber bundle played an essential role and there might exist different interpolations leading to different topological charges. But such is not the case. The topological information of the underlying lattice is carried by the gauge potentials and the value of the topological charge is completely unique for a given gauge field configuration. In order to see that the topological charge (5.5) is indeed an integer given by (5.14) no interpolation to a smooth bundle is necessary. This way of understanding, however, makes the topological meaning of (5.5) obscure. So it is more desirable to have a formalism of topological invariants of fiber bundles over discrete lattices without leaning upon the interpolation method.

What is the implication of the present analysis on the index theorem of the Ginsparg-Wilson Dirac operator on the lattice? We infer that the topological charge (5.5) coincides with the index of the Ginsparg-Wilson Dirac operator on the periodic lattice Λ as the index theorem suggests. In the case of infinite lattice the chiral anomaly coincides with the Chern character. However, both the index and the topological charge are not well-defined in general. On the other hand both of them are well-defined in the case of finite lattice. Unfortunately, we have very little known about the precise connection between them. The situation gets even worse for nonabelian theories.^{††} The topological charges obtained so far in the literature by interpolations [8, 10, 11] are too complicated to manipulate and no differential geometric construction of Chern characters leading to topological invariants seems to be available on the lattice. We must extend the theorem given in the Introduction to nonabelian theories in order to classify the topological invariants on the lattice. It is a challenging problem to establish the lattice extension of the index theorem.

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^{††}As a related study, see ref. [14].

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